A BRIEF NOTE ON SETS

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Let X be a set, $A, B \subset X$. We can trivially rewrite these sets $A = \{x \in X; x \in A\}$, $B = \{x \in X; x \in B\}$. Dropping the reference to the universal set, $A = \{x \in A\}$, $B = \{x \in B\}$. We can also write $A \cap B = \{x \in A\} \cap \{x \in B\} = \{x \in A \cap B\}$. And the same applies to unions, complements. So we have a nice little lexical homomorphism going here.

Let Y be another set, with $F : Y \to X$. We can elevate F to a set-function from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ (the power sets of X and Y respectively) by the canonical rule $F(S) = \cup_{y \in S}\{F(y)\}$. Defining the preimage[1] accordingly, $F^{-1}(A) = \{y \in Y; F(y) \in A\}$. Or again dropping the universal set, $F^{-1}(A) = \{F \in A\}$. It's a basic result on preimages that $F^{-1}(A \cap B) = F^{-1}(A) \cap F^{-1}(B)$. And so we have $\{F \in A\} \cap \{F \in B\} = \{F \in A \cap B\}$. And the same for unions, complements.

Let Σ be a σ-algebra of subsets of X. We can elevate F again to a function from $\mathcal{P}^2(X)$ to $\mathcal{P}^2(Y)$ and write $F(\Sigma) = \cup_{S \in \Sigma}\{F(S)\}$. Again we can define a preimage $F^{-1}(\Sigma) = \{F \in \Sigma\}$ and we can easily check that $F^{-1}(\Sigma)$ is a σ-algebra on Y. Note that $F^{-1}(\Sigma) = \sigma(F)$, the σ-algebra generated by F. It is the smallest σ-algebra on Y that makes F measurable. The same arguments apply to topologies; only the vocabulary changes. If $\mathcal{O}$ is a topology on X then $F^{-1}(\mathcal{O})$ is the initial topology on Y with respect to F.

At this level, we can also elevate F to a family of functions $\mathfrak{F}$ defined by $\mathfrak{F}(\Sigma) = \cup_{F \in \mathfrak{F}}\{F(\sigma)\}$. We can then define the σ-algebra generated by $\mathfrak{F}$ to be the smallest σ-algebra containing $\mathfrak{F}^{-1}(\Sigma) = \{\mathfrak{F} \in \Sigma\}$. Of course we must do a little work to prove that this σ-algebra exists and is unique (it does and it is). We can likewise extend the definition of the initial topology. Observe that, while we could have made a similar function-family extension to our set-function framework $\mathcal{P}$, it only becomes interesting at the set-set-function level with the accompanying structure (i.e. σ-algebras and topologies) of the domain.

What happens if we continue elevating? What can we say about $F : \mathcal{P}^3(X) \to \mathcal{P}^3(Y)$? It now seems natural to talk about filtrations. Let $\mathbb{G} = (G_k)_{k \geq 0}$ be a filtration and let's remain agnostic for the moment about the underlying algebraic structure. Define $F(\mathbb{G}) = (F(G_k))_{k \geq 0}$. Because that the subset relationship between the elements of $\mathbb{G}$ implies the index set, we can drop the indices and simply write $F(\mathbb{G}) = \cup_{G \in \mathbb{G}}\{G\}$, which is nicely consistent with our definitions above. Note that $F(\mathbb{G})$ is also a filtration, as is $F^{-1}(\mathbb{G})$.

I'm going to stop here. Sorry there's no punchline. But it's all vaguely algebraic and smells a bit like category theory so I figure I'll pass this along.

[1]While a function doesn't always have an inverse, a set-function always will. This reminds of me of how functions don't always have derivatives, but once we elevate them to generalized functions they do. Seems like there's a common pattern there. I'm also reminded of the concept of null values in computer science.