QUOTIENT σ-ALGEBRAS

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Let X and Y be sets with some map $f : X \to Y$. We can identify $x, y \in X$ by the equivalence relation $x \sim y$ iff $f(x) = f(y)$. This relation is called the kernel of f. Consider the quotient space $X/\sim$, or alternatively $X/\ker f$. This set-theoretic construction is pretty boring and just amounts to a set of partitions. It becomes more interesting when we add more structure to X and Y, which manifests when quotienting topological spaces, or linear spaces, or groups. In what follows we will explore the quotient of a measurable space.

Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces and let $f : X \to Y$ be a measurable function. Our first obstacle to a measure-theoretic quotient is that our set-theoretic quotient looks covariant, whereas measurability is contravariant. But we can rephrase our set quotient contravariantly: we redefine $x \sim y$ to hold iff $x, y \in f^{-1}(a)$ for some $a \in Y$. That is $x \sim y$ when they are in the same fiber under f.

We can now identify members $A, B \in \mathcal{F}$ through the parallel construction $A \sim B$ iff $A, B \in f^{-1}(G)$ for some $G \in \mathcal{G}$. Here we define f^{-1} to be the elements of $\mathcal{F}$ that map to G, rather than the more standard definition as a set of points. We define the quotient σ-algebra $\mathcal{F}/\ker f$ where the kernel is now understood as a partition of $\mathcal{F}$. Suppose $F_n \in \mathcal{F}/\ker f$ for a countable collection F_n. Then for some $G_n \in \mathcal{G}$, $f(A) = G_n$ for each $A \in F_n$. Because $\mathcal{G}$ is a σ-algebra, $\cup G_n = G$ for some $G \in \mathcal{G}$ and we define $F = f^{-1}(G)$. Then $F = f^{-1}(G) = f^{-1}(\cup G_n) = \cup f^{-1}(G_n) = \cup F_n$. The same holds for complements and we conclude that $\mathcal{F}/\ker f$ is a σ-algebra.

The preceeding construction of $\mathcal{F}/\ker f$ is a σ-algebra built over partitions of X, whereas $\mathcal{F}$ is built over X. Like we can embed $\mathbb{Z}/n\mathbb{Z}$ into $\mathbb{Z}$, we can likewise embed $\mathcal{F}/\ker f$ into $\mathcal{F}$. Unlike the integers, we can do this canonically by mapping $F \in \mathcal{F}/\ker f$ to $\cup F_i$. Observe that $\cup F_i \in F$ because $f(\cup F_i) = \cup f(F_i)$ and because $\ker f$ is a partition this map is injective. This embedding is better known as the σ-algebra of f, $\sigma(f)$. We have also motivated the standard definition of $f^{-1}(G)$; we identify this collection of sets with its canonical representative.

For example, let $(X, \mathcal{F}) = (X_1 \times X_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ where $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$ are measurable spaces. Define $f : (X, \mathcal{F}) \to (X_1, \mathcal{F}_1)$ by the relation $f(x) = f(x_1, x_2) = x_1$. Clearly f is measurable. The quotient set $X/\ker f$ is sets of the form $\{(x_1, x_2); x_2 \in X_2\}$ for fixed $x_1 \in X_1$. Clearly $X/\ker f \cong X_1$, but this isomorphism is not canonical. In particular, we have to pick some arbitrary x_2 to be the representative of each equivalence class. The quotient σ-algebra $\mathcal{F}/\ker f$ is sets of the form $\{F; F \in f^{-1}(F_1)\}$ for some $F_1 \in \mathcal{F}_1$. And again it is clear that $\mathcal{F}/\ker f \cong \mathcal{F}_1$ but this time the isomorphism is canonical. In particular, we define the map using the canonical representative $\cup f^{-1}(F_1)$.