CHANGE OF VARIABLES

JOHN THICKSTUN

Let $f : \mathbb{R} \to \mathbb{R}$ be the simple quadratic function $f(x) = x^2$. The fundamental theorem of calculus tells us that

$$\int_{-1}^{1} x^2 dx = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

Alternatively, we can make a naive substitution $u = x^2$. Some formal manipulations give us $du = 2xdx$ and therefore $dx = \frac{du}{2x} = \frac{du}{2\sqrt{u}}$. Changing the limits of integration, we have

$$\int_{-1}^{1} x^2 dx = \int_{1}^{1} \frac{u}{2\sqrt{u}} du = \frac{1}{2}\int_{1}^{1} \sqrt{u} du.$$

Integrating over a single point is just zero, which implies that the right-hand side of the above integral vanishes. But our earlier calculation shows that this is clearly not true; something has gone wrong. The problem is that our substituting function is not injective. Let's try again, this time splitting our integral into two segments for which u is injective before making the u-substitution:

$$\int_{-1}^{1} x^2 dx = \int_{-1}^{0} x^2 dx + \int_{0}^{1} x^2 dx = \frac{1}{2}\int_{1}^{0} \sqrt{u} du + \frac{1}{2}\int_{0}^{1} \sqrt{u} du.$$

$$= -\frac{1}{2}\int_{0}^{1} \sqrt{u} du + \frac{1}{2}\int_{0}^{1} \sqrt{u} du = 0.$$

So we're still in trouble. Now the problem is that we took the positive square root of u. That's ok, in fact the general change of variables formula tells us to do this, but we need to be careful about the orientation of our domain. So let's back up again and clarify our domain of integration

$$\int_{-1}^{1} x^2 dx = \int_{[-1,1]} x^2 dx = \int_{[-1,0)} x^2 dx + \int_{[0,1]} x^2 dx$$

$$= \frac{1}{2}\int_{u([-1,0))} \sqrt{u} du + \frac{1}{2}\int_{u([0,1])} \sqrt{u} du = \frac{1}{2}\int_{(0,1]} \sqrt{u} du + \frac{1}{2}\int_{[0,1]} \sqrt{u} du$$

$$= \int_{0}^{1} \sqrt{u} du = \frac{2}{3}u^{3/2}\Big|_{0}^{1} = \frac{2}{3}.$$

Stripping the orientation from our domain of integration cancels with taking the positive square root, so now we are good!

Let's have some more fun. Instead of integrating over $[-1,1]$, which is pretty boring, let's integrate over the half-circle $C = \{e^{i\theta} \in \mathbb{C}; 0 \leq \theta \leq \pi\}$. We can pull this curve back to $\mathbb{R}$ with the parameterization $c(\theta) = e^{i\theta}$ where $\theta \in [0,\pi]$ and write

$$\int_C x^2 dx = \int_c x^2 dx = \int_{[0,\pi]} c(\theta)^2 c'(\theta) d\theta = i \int_{[0,\pi]} e^{2i\theta} e^{i\theta} d\theta$$

$$= i \int_{[0,\pi]} e^{3i\theta} d\theta = \frac{i}{3i} e^{3i\theta} \Big|_{pi}^{0} = \frac{2}{3}.$$

Of course, we could have predicted this result from Cauchy's integral theorem.