CHANGE OF MEASURE

JOHN THICKSTUN

Suppose P is be a σ-finite measure and X is a r.v. on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{B}(\mathbb{R})$ and $\mathcal{L}(\mathbb{R})$ denote the Borel and Lebesgue σ-algebras respectively. We can define the pushforward measure $X_*P : \mathcal{L}(\mathbb{R}) \to \mathcal{B}(\mathbb{R})$ for any $B \in \mathcal{L}(\mathbb{R})$ by the map

$$X_*P(B) = P(X \in B) = \int_\Omega \mathbb{1}_{\{X \in B\}} dP.$$

This map is more commonly called the **law** of X, often denoted P^X. Because X_*P is itself a measure, we can naturally write

$$X_*P(B) = \int_B d(X_*P).$$

We are now in a position to state a natural change of variable formulas (obviously we must do some work to prove that the equation holds; it conveniently does whenever the integrands are integrable)

$$\int_\Omega f \circ X dP = \int_\mathbb{R} f d(X_*P).$$

This is, of course, just the familiar law of the unconscious statistician:

$$\mathbb{E}\{f(X)\} = \int_\Omega f(X) dP = \int_{-\infty}^{\infty} f(x) d(X_*P).$$

Using this theorem with an indicator function, we can rewrite probabilities as expectations:

$$P(X \in B) = \int_B d(X_*P) = \int_{-\infty}^{\infty} \mathbb{1}_{x \in B} d(X_*P) = \int_\Omega \mathbb{1}_{\{X \in B\}} dP = \mathbb{E}\{\mathbb{1}_{\{X \in B\}}\}.$$

Let μ be Lebesgue measure. If $X_*P \ll \mu$ then by the Radon-Nikodym theorem $dX_*P/d\mu$ exists and

$$P(X \in B) = X_*P(B) = \int_B \frac{dX_*P}{d\mu} d\mu.$$

This Radon-Nikodym derivative is more commonly referred to as the **density** of X; note the crucial requirement of absolute continuity which in this case (necessarily) exactly coincides with the existence of a density. By the chain rule,

$$E\{f(X)\} = \int_{-\infty}^{\infty} f(x) d(X_*P) = \int_{-\infty}^{\infty} f(x) \frac{dX_*P}{d\mu} d\mu,$$

$$P(X \in B) = \int_{-\infty}^{\infty} \mathbb{1}_{x \in B} d(X_*P) = \int_{-\infty}^{\infty} \mathbb{1}_{x \in B} \frac{dX_*P}{d\mu} d\mu.$$

Let Q another σ-finite measure on $(\Omega, \mathcal{F})$. By the Radon-Nikodym theorem, there is some $\mathcal{F}$-measurable function dQ/dP such that for all $A \in \Omega$,

$$Q(A) = \int_A \frac{dQ}{dP} dP.$$

We can compute the pushforward measure of Q in terms of P by the chain rule:

$$Q(X \in B) = X_* Q(B) = \int_\Omega \mathbb{1}_{\{X \in B\}} dQ = \int_\Omega \mathbb{1}_{\{X \in B\}} \frac{dQ}{dP} dP = \mathbb{E}_P \left\{ \frac{dQ}{dP} \mathbb{1}_{\{X \in B\}} \right\}.$$

And similarly for expectations,

$$\mathbb{E}_Q\{f(X)\} = \int_\Omega f(X) dQ = \int_\Omega f(X) \frac{dQ}{dP} dP = \mathbb{E}_P \left\{ \frac{dQ}{dP} f(X) \right\}.$$

This is as much as we can say in general. But now consider the special case that P and Q are measures on $(\mathbb{R}, \mathcal{L}(\mathbb{R}))$. Such a situation may arise for example, if P and Q are pushforward measures. Then by the chain rule,

$$Q(X \in A) = \int_A dQ = \int_A \frac{dQ}{dP} dP = \int_A \frac{dQ}{dP} \frac{dP}{d\mu} d\mu.$$

And similarly we have

$$\mathbb{E}_Q\{f(X)\} = \int_\mathbb{R} f(X) \frac{dQ}{dP} \frac{dP}{d\mu} d\mu.$$

These results seem very nice and intuitive. Change of measure is just change of variables, and we can "fix up" a change of measure by multiplying in the derivative.