NEGATIVE PROBABILITIES AND NUMERICS IN THE CRR OPTION MODEL

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1. Negative Probabilities

Recall the definition of the risk neutral probability π in the binomial option pricing model. Let r_n be the single period risk-free rate, u,d be the up and down parameters of the general binomial model, and n be the number of branches in the binomial tree. In the particular case of the CRR model, we control u and d with a single hyperparameter σ, intended to represent volatility, and set $u, d = \exp(\pm\sigma\sqrt{T/n})$. The risk neutral probability is then given by

$$\pi = \frac{r_n - d}{u - d} = \frac{r_n - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}}$$

Note that there is nothing in this equation that enforces $\pi \in [0, 1]$. In particular, $\pi \in [0, 1]$ if and only if $r_n \in [d, u]$. Note that we *do* require that the parameters $\sigma, T, n > 0$ and therefore $d < 1$. As long as the interest rate is not negative, we will therefore always have $d < r_n$. But the bound on u is more interesting. Many otherwise valid combinations of σ, T, n will produce a value $u < r_n$. In particular suppose $T = n = 1$, the simple 1-step binomial tree. Then $u = \exp(\sigma) \approx 1 + \sigma$ for sufficiently small σ. For example, if $\sigma = .01$ and $r_n = 1.02$ then

$$\pi = \frac{r_n - \exp(-.01)}{\exp(.01) - \exp(-.01)} \approx \frac{1.02 - .99}{1.01 - .99} = 1.5.$$

Note that if $\pi = 1.5$ then $1 - \pi = -.5$, hence the labelling of this phenomenon as "negative probability." This result is not so counterintuitive when we consider that the risk neutral probability π is a probability only in name. By definition, it is the measure under which the discounted expectation of the binomial outcomes equals the arbitrage enforced price. There is no a-priori reason to believe that this measure is a probability measure, or even to believe that it exists or is unique.[1]

Qualitatively, we will encounter negative probabilities when σ is relatively small and the risk-free rate is relatively high. Quantitatively, we have the following proposition.

[1] I think the fundamental theorem of asset pricing states that existence is guaranteed by non-existence of arbitrage opportunities, and uniqueness by market completeness. But I don't know enough to say this with confidence yet.

Proposition 1.1. *The value* $1-\pi$ *is negative exactly when*

$$\sigma < \log(r_n)\sqrt{n/T}.$$

Proof. Recall the definition of $1-\pi$:

$$1-\pi = 1 - \frac{r_n - d}{u-d} = \frac{u-r_n}{u-d}.$$

Note by the CRR definitions of u and d that $u-d>0$. Therefore $1-\pi<0$ iff $u<r_n$. Appealing again to the CRR definition of u and solving for σ gives us our result. □

We could also ask whether tinkering with the other model parameter, the number of steps n, helps. It can, but the relationship runs inversely to (the square root of) the number of steps. I.e. As we grow the size of our binomial tree, we are more likely to encounter negative probabilities.

Proposition 1.2. *Asymptotically, as the number of steps* n *grows,*

$$\log(r_n)\sqrt{n/T} = O(1/\sqrt{n}).$$

Proof. By definition of the period risk-free rate,

$$\log(r_n)\sqrt{n/T} = \log(e^{rT/n})\sqrt{n/T} = \frac{rT}{n}\sqrt{n/T} = O\left(\frac{1}{\sqrt{n}}\right).$$ □

Note by the same argument that $\log(r_n)\sqrt{n/T} = O(\sqrt{T})$. So we are more likely to encounter negative probabilities for short dated options.

2. Numerical Considerations

This problem comes up in practice. I once built an implied volatility solver on top of a CRR binomial pricer. The generic numerical solver proceeds in two steps. First, it brackets a root starting from a guess by expanding an interval around the guess until its boundary points have different sign (there's also a second entry into the solver that skips this step, allowing the user to explicitly specify an interval containing the root). Then it uses the Brent solver on this interval to identify the root.

There are two challenges for this generic solver in resolving CRR implied volatilities. The first, as we have already anticipated, is that it can fail in the root bracketing step by testing a left bound on its bracket that is too small, resulting in negative probabilities in the pricer. I worked around this problem by progressively attempting to bracket with lower bounds of .5%, 1%, and 2% respectively. But the analysis in section 1 indicates that the correct lower bound ought to be exactly $\log(r_n)\sqrt{n/T}$. The solver could still fail to resolve, but in this case the failure is entirely due to limitations of the CRR model, and not an artifact of the solver methodology.

The second potential challenge for the numerical solver is that the CRR prices are estimates. In particular, they can fail to be monotonic in volatility due to limited accuracy.

This is a theoretical problem for the pricer, potentially preventing convergence to an implied volatility. In practice, I have not seen this happen. But if it becomes a problem, we could circumvent it with an analysis similar to the one we made above for negative probabilities.

Controlling the numerical accuracy of the CRR prices would proceed as follows. Intuitively, while CRR prices are not monotonic, they are *almost* monotonic. Therefore while the solver might fail to resolve due to CRR error, the mode of failure is that the solver would bounce endlessly between volatility estimates that are very close to the true implied volatility, but not quite of the desired accuracy. I expect it is possible to derive an analytic expression for the CRR error, or at least a tight bound on it. We could then set the desired accuracy of the solver to this error, which would yield implied volatility estimates at the maximum level of accuracy possible given the CRR model.